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Relation Between Sums of Squares and Sums of Centred Pentagonal Numbers Induced by Partitions of 8

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ABSTRACT

The relation between the number of representations of an integer n as a sums of squares induced by λ, $s_\lambda(n)$ and the number of representations of an integer n as a sums of centred pentagonal numbers induced by λ, $c_\lambda(n)$ is given by

$$s_\lambda(8n) - s_\lambda(2n) = \beta_\lambda c_\lambda(5n+3),$$

for all non-negative integers n where $\beta_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right)$ for i_j denote the number of parts of λ which are equal to j and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a partition of 8. This relation is proved for all 22 partitions of 8 by using the generating function method.

Keywords: Number of representations, squares, centred pentagonal numbers.

1. Introduction

A relation between number of representations of integer n as a sums of k squares, $s_k(n)$ and number of representations of integer n as a sums of k triangular numbers, $t_k(n)$ has been discovered by Barrucand et al. (2002) given by

$$s_k(8n+k) = \alpha_k t_k(n),$$

where

$$\alpha_k = 2^{k-1}\left\{2+\binom{k}{4}\right\},\ for\ 1 \leq k \leq 7.$$

Generating functions method was used in proving this relation. Later, this relation was proved using the combinatorial method by Cooper and Hirschhorn (2004). However this relation does not hold for any $k \geq 8$. This fact has been proved by Bateman et al. (2001).

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition of k where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are integers satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$ and $\lambda_1+\lambda_2+\ldots+\lambda_m = k$. We denote $s_\lambda(n)$ the number of representations of an integer n, a sum of squares induced by λ and $t_\lambda(n)$ the number of representations of an integer n as a sum of triangular number induced by λ. In other words, $s_\lambda(n)$ is the number of integer solutions of the equation

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_m x_m^2 = n$$

and $t_\lambda(n)$ is the number of non-negative integer solutions in the equation

$$\lambda_1 \frac{x_1(x_1-1)}{2} + \lambda_2 \frac{x_2(x_2-1)}{2} + \ldots + \lambda_m \frac{x_m(x_m-1)}{2} = n.$$

Adiga et al. (2005) obtained a relation between $s_\lambda(n)$ and $t_\lambda(n)$ as

$$s_\lambda(8n+k) = \beta_\lambda t_\lambda(n)$$

for $1 \leq k \leq 7$, where

$$\beta_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right) \tag{1}$$

for i_j denote the number of parts in λ which are equal to j.

They proved the result using the generating functions and combinatorial method. Baruah et al. (2008) proved the relationship between $s_\lambda(n)$ and $t_\lambda(n)$ given by

$$s_\lambda(8n+8) - s_\lambda(2n+2) = \beta_\lambda t_\lambda(n)$$

for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition of 8 using generating function method.

Let $c_\lambda(n)$ denote the number of representations of an integer n as a sums of centred pentagonal numbers induced by λ. In other words, $c_\lambda(n)$ is the number of non-negative integer solutions of the equation

$$\lambda_1 \frac{5x_1^2 + 5x_1 + 2}{2} + \lambda_2 \frac{5x_2^2 + 5x_2 + 2}{2} + \ldots + \lambda_m \frac{5x_m^2 + 5x_m + 2}{2} = n.$$

For example, for $n = 23$ and $\lambda = (5, 2, 1)$ we have $m = 3$,

$$5\frac{5x_1^2 + 5x_1 + 2}{2} + 2\frac{5x_2^2 + 5x_2 + 2}{2} + \frac{5x_3^2 + 5x_3 + 2}{2} = 23.$$

Then $23 = 5(1)+2(6)+1(6) = 5(1)+2(1)+1(16)$. Thus, $(x_1, x_2, x_3) = (0, 1, 1) = (0, 0, 2)$. Therefore $c_{(5,2,1)}(23) = 2$.

Johari et al. (2012) obtained the relation between $s_\lambda(n)$ and $c_\lambda(n)$ as

$$s_\lambda\left(\frac{8n-3k}{5}\right) = \beta_\lambda c_\lambda(n)$$

for $1 \leq k \leq 7$ where β_λ is given by Equation (1) and they proved for this relation by using the generating function method. Later, the combinatorial proof was given by Johari et al. (2013).

Ballantine and Merca (2019) used Merca's factorization theorem for Lambert series to obtain relationships between $s_k(n)$ and partitions into distinct parts. They also obtained convolutions involving overpartition functions as well as pentagonal recurrence formulas for $s_4(n)$ and $s_8(n)$. These results lead to new connections between divisors and partitions.

Jha (2020) established a relation between sum of inverses of odd divisors d of a positive integer n and the number of representations of n as a sum of k squares $s_k(n)$. They proved the following interesting combinatorial identity:

$$\sum_{d|n} \frac{1}{d} = \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^{n+k}}{k} \binom{n}{k} s_k(n).$$

2. A relation between sum of squares and sums of centred pentagonal numbers induced by partitions of 8

In this section, we extend our discussion to the relation between the number of representations of an integer n as sums of squares and number of representations of n as sums of centred pentagonal numbers induced by partitions of 8. The relation is given by following theorem.

Theorem 1. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition of 8. Then for any non-negative integer n, we have*

$$s_\lambda(8n) - s_\lambda(2n) = \beta_\lambda c_\lambda(5n+3),$$

where $\beta_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right)$ and i_j denote the number of parts in λ which are equal to j.

We prove Theorem 1 by using generating function method. Let

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \psi(q) = \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}}, \omega(q) = \sum_{n=0}^{\infty} q^{\frac{5n^2+5n+2}{2}},$$

where q is any number in $(-1,1)$. The generating functions for $s_\lambda(n)$ and $c_\lambda(n)$ are

$$\sum_{n=0}^{\infty} s_{(\lambda)}(n) q^n = \phi(q^{\lambda_1})\phi(q^{\lambda_2})\ldots\phi(q^{\lambda_m}),$$

$$\sum_{n=0}^{\infty} c_{(\lambda)}(n) q^n = \omega(q^{\lambda_1})\omega(q^{\lambda_2})\ldots\omega(q^{\lambda_m}).$$

In order to prove Theorem 1, we need the following lemma.

Lemma 2.1. *Let*

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \psi(q) = \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}}, \omega(q) = \sum_{n=0}^{\infty} q^{\frac{5n^2+5n+2}{2}},$$

$$a(q) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

where q is any number in $(-1,1)$. *Then, we have the following relations*

$$\phi(q)+\phi(-q) = 2\phi\left(q^4\right), \tag{2}$$
$$\phi(q)-\phi(-q) = 4q\psi\left(q^8\right), \tag{3}$$
$$\phi(q)\phi(-q) = \phi^2\left(-q^2\right), \tag{4}$$
$$\phi(q)\psi\left(q^2\right) = \psi^2(q), \tag{5}$$
$$a(q) = \phi(q)\phi\left(q^3\right)+4q\psi\left(q^2\right)\psi\left(q^6\right), \tag{6}$$
$$\phi(q) = \phi\left(q^4\right)+2q\psi\left(q^8\right), \tag{7}$$
$$\phi^2(q) = \phi^2\left(q^2\right)+4q\psi^2\left(q^4\right), \tag{8}$$
$$\psi(q)\psi\left(q^3\right) = \phi\left(q^6\right)\psi\left(q^4\right)+q\phi\left(q^2\right)\psi\left(q^{12}\right), \tag{9}$$
$$\phi(q)\phi\left(q^3\right) = a\left(q^4\right)+2q\psi\left(q^2\right)\psi\left(q^6\right), \tag{10}$$
$$a(q) = a\left(q^4\right)+6q\psi\left(q^2\right)\psi\left(q^6\right), \tag{11}$$
$$\psi(q) = q^{-\frac{1}{5}}\omega\left(q^{\frac{1}{5}}\right). \tag{12}$$

Proof. The proof of the Equations (2)-(11) is discussed in Baruah et al. (2008). We provide the proof for Equation (12) as follows.

$$\begin{aligned}\psi(q) &= \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}} \\ &= \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}+\frac{1}{5}-\frac{1}{5}} \\ &= \sum_{n=0}^{\infty} q^{\frac{5n^2+5n+2}{10}-\frac{1}{5}} \\ &= q^{-\frac{1}{5}}\sum_{n=0}^{\infty} q^{\left(\frac{1}{5}\right)\left(\frac{5n^2+5n+2}{2}\right)} \\ &= q^{-\frac{1}{5}}\omega\left(q^{\frac{1}{5}}\right).\end{aligned}$$

Thus, Equation (12) is proved.

□

Now, by using the Lemma 2.1, we prove the Theorem 1.

Proof of Theorem 1

There are 22 partitions to be considered in this proof. However we only show two partitions which are $(5,1,1,1)$ and $(3,3,1,1)$. Similar approach is used for proving other 20 partitions.

1. Case $\lambda = (5,1,1,1)$

$$\sum_{n=0}^{\infty} s_{(5,1,1,1)}(n)q^n = \phi\left(q^5\right)\phi\left(q\right)\phi^2\left(q\right).$$

Apply Equation (7) and Equation (8) to get

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(5,1,1,1)}(n)q^n &= \left[\phi\left(q^{20}\right)+2q^5\psi\left(q^{40}\right)\right]\left[\phi\left(q^4\right)+2q\psi\left(q^8\right)\right]\\ &\quad \left[\phi^2\left(q^2\right)+4q\psi^2\left(q^4\right)\right]\\ &= \phi\left(q^{20}\right)\phi\left(q^4\right)\phi^2\left(q^2\right)+4q\phi\left(q^{20}\right)\phi\left(q^4\right)\psi^2\left(q^4\right)\\ &\quad +2q\phi\left(q^{20}\right)\psi\left(q^8\right)\phi^2\left(q^2\right)+8q^2\phi\left(q^{20}\right)\psi\left(q^8\right)\psi^2\left(q^4\right)\\ &\quad +2q^5\psi\left(q^{40}\right)\phi\left(q^4\right)\phi^2\left(q^2\right)+8q^6\psi\left(q^{40}\right)\phi\left(q^4\right)\psi^2\left(q^4\right)\\ &\quad +4q^6\psi\left(q^{40}\right)\psi\left(q^8\right)\phi^2\left(q^2\right)+16q^7\psi\left(q^{40}\right)\psi\left(q^8\right)\psi^2\left(q^4\right).\end{aligned}$$

Extract the even powers of q, then replace q^2 with q, gives

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(5,1,1,1)}\left(2n\right)q^n &= \phi\left(q^{10}\right)\phi\left(q^2\right)\phi^2\left(q\right)+8q\phi\left(q^{10}\right)\psi\left(q^4\right)\psi^2\left(q^2\right)\\ &\quad +8q^3\psi\left(q^{20}\right)\phi\left(q^2\right)\psi^2\left(q^2\right)\\ &\quad +4q^3\psi\left(q^{20}\right)\psi\left(q^4\right)\phi^2\left(q\right). \end{aligned} \tag{13}$$

Apply Equation (8) to Equation (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(5,1,1,1)}\left(2n\right)q^n &= \phi\left(q^{10}\right)\phi\left(q^2\right)\left[\phi^2\left(q^2\right)+4q\psi^2\left(q^4\right)\right]\\ &\quad +8q\phi\left(q^{10}\right)\psi\left(q^4\right)\psi^2\left(q^2\right)+8q^3\psi\left(q^{20}\right)\phi\left(q^2\right)\psi^2\left(q^2\right)\\ &\quad +4q^3\psi\left(q^{20}\right)\psi\left(q^4\right)\left[\phi^2\left(q^2\right)+4q\psi^2\left(q^4\right)\right].\end{aligned}$$

Extract the even powers of q, then replace q^2 with q, yields

$$\sum_{n=0}^{\infty} s_{(5,1,1,1)}\left(4n\right)q^n = \phi\left(q^5\right)\phi\left(q\right)\phi^2\left(q\right)+16q^2\psi\left(q^{10}\right)\psi^3\left(q^2\right). \tag{14}$$

Apply Equation (7) and Equation (8) to Equation (14) , we have

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(5,1,1,1)}(4n)\, q^n &= \left[\phi\left(q^{20}\right)+2q^5\psi\left(q^{40}\right)\right]\left[\phi\left(q^4\right)+2q\psi\left(q^8\right)\right] \\ &\quad \left[\phi^2\left(q^2\right)+4q\psi^2\left(q^4\right)\right]+16q^2\psi\left(q^{10}\right)\psi^3\left(q^2\right) \\ &= \phi\left(q^{20}\right)\phi\left(q^4\right)\phi^2\left(q^2\right)+4q\phi\left(q^{20}\right)\phi\left(q^4\right)\psi^2\left(q^4\right) \\ &\quad +2q\phi\left(q^{20}\right)\psi\left(q^8\right)\phi^2\left(q^2\right)+8q^2\phi\left(q^{20}\right)\psi\left(q^8\right)\psi^2\left(q^4\right) \\ &\quad +2q^5\psi\left(q^{40}\right)\phi\left(q^4\right)\phi^2\left(q^2\right)+8q^6\psi\left(q^{40}\right)\phi\left(q^4\right)\psi^2\left(q^4\right) \\ &\quad +4q^6\psi\left(q^{40}\right)\psi\left(q^8\right)\phi^2\left(q^2\right)+16q^7\psi\left(q^{40}\right)\psi\left(q^8\right)\psi^2\left(q^4\right) \\ &\quad +16q^2\psi\left(q^{10}\right)\psi^3\left(q^2\right).\end{aligned}$$

Extract the even powers of q, then replace q^2 with q, the result is

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(5,1,1,1)}(8n)\, q^n &= \phi\left(q^{10}\right)\phi\left(q^2\right)\phi^2(q)+8q\phi\left(q^{10}\right)\psi\left(q^4\right)\psi^2\left(q^2\right) \\ &\quad +8q^3\psi\left(q^{20}\right)\phi\left(q^2\right)\psi^2\left(q^2\right)+4q^3\psi\left(q^{20}\right)\psi\left(q^4\right)\phi^2(q) \\ &\quad +16q\psi\left(q^5\right)\psi^3(q).\end{aligned}$$

From Equation (13), we have

$$\sum_{n=0}^{\infty} s_{(5,1,1,1)}(8n)\, q^n = \sum_{n=0}^{\infty} s_{(5,1,1,1)}(2n)\, q^n+16q\psi\left(q^5\right)\psi^3(q). \qquad (15)$$

Apply Equation (12) to Equation (15), we have

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(5,1,1,1)}(8n)\, q^n &= \sum_{n=0}^{\infty} s_{(5,1,1,1)}(2n)\, q^n+16q\left[q^{-\frac{5}{5}}\omega\left(q^{\frac{5}{5}}\right)\right]\left[q^{-\frac{1}{5}}\omega\left(q^{\frac{1}{5}}\right)\right]^3 \\ &= \sum_{n=0}^{\infty} s_{(5,1,1,1)}(2n)\, q^n+16q^{-\frac{3}{5}}\omega\left(q^{\frac{5}{5}}\right)\omega^3\left(q^{\frac{1}{5}}\right) \\ &= \sum_{n=0}^{\infty} s_{(5,1,1,1)}(2n)\, q^n+16\sum_{n=0}^{\infty} c_{(5,1,1,1)}(n)\, q^{\frac{n-3}{5}}.\end{aligned}$$

Equate coefficients of q^n, we get

$$s_{(5,1,1,1)}(8n) = s_{(5,1,1,1)}(2n)+16c_{(5,1,1,1)}(5n+3).$$

For $\lambda=(5,1,1,1)$, we have $m=4, i_1=3, i_2=0, i_3=0$. Hence

$$\beta_{(5,1,1,1)}=2^4+2^3\left(\binom{3}{4}+\binom{3}{2}\binom{0}{1}+\binom{3}{1}\binom{0}{1}\right)=2^4=16.$$

Thus, Theorem 1 for the partition $\lambda = (5,1,1,1)$ is proved.

2. Case $\lambda = (3,3,1,1)$

$$\sum_{n=0}^{\infty} s_{(3,3,1,1)}(n)\, q^n = \phi\left(q^3\right)\phi\left(q^3\right)\phi\left(q\right)\phi\left(q\right)$$

Apply Equation (10), we have

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(3,3,1,1)}(n)\, q^n &= \left[a\left(q^4\right)+2q\psi\left(q^2\right)\psi\left(q^6\right)\right]\left[a\left(q^4\right)+2q\psi\left(q^2\right)\psi\left(q^6\right)\right] \\ &= a^2\left(q^4\right)+4qa\left(q^4\right)\psi\left(q^2\right)\psi\left(q^6\right)+4q^2\psi^2\left(q^2\right)\psi^2\left(q^6\right).\end{aligned}$$

Extract the even powers of q, then replace q^2 with q. The result is

$$\sum_{n=0}^{\infty} s_{(3,3,1,1)}(2n)\, q^n = a^2\left(q^2\right)+4q\psi^2\left(q\right)\psi^2\left(q^3\right). \tag{16}$$

Apply Equation (9) and Equation (6) to Equation (16) , we have

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(3,3,1,1)}(2n)\, q^n &= \left[\phi\left(q^2\right)\phi\left(q^6\right)+4q^2\psi\left(q^4\right)\psi\left(q^{12}\right)\right]^2 \\ &\quad +4q\left[\phi\left(q^6\right)\psi\left(q^4\right)+q\phi\left(q^2\right)\psi\left(q^{12}\right)\right]^2 \\ &= \phi^2\left(q^2\right)\phi^2\left(q^6\right)+16q^4\psi^2\left(q^4\right)\psi^2\left(q^{12}\right) \\ &\quad +4q\phi^2\left(q^6\right)\psi^2\left(q^4\right)+16q^2\phi\left(q^6\right)\psi\left(q^4\right)\phi\left(q^2\right)\psi\left(q^{12}\right) \\ &\quad +4q^3\phi^2\left(q^2\right)\psi^2\left(q^{12}\right).\end{aligned}$$

Extract the even powers of q, then replace q^2 with q yields

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(3,3,1,1)}(4n)\, q^n &= \phi^2\left(q\right)\phi^2\left(q^3\right)+16q^2\psi^2\left(q^2\right)\psi^2\left(q^6\right) \\ &\quad +16q\phi\left(q\right)\phi\left(q^3\right)\psi\left(q^2\right)\psi\left(q^6\right).\end{aligned} \tag{17}$$

Apply Equation (10) to Equation (17) , we have

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(3,3,1,1)}(4n)\,q^n &= \left[a\left(q^4\right)+2q\psi\left(q^2\right)\psi\left(q^6\right)\right]^2\\ &\quad +16q^2\psi^2\left(q^2\right)\psi^2\left(q^6\right)+16q\psi\left(q^2\right)\psi\left(q^6\right)\\ &\quad \left[a\left(q^4\right)+2q\psi\left(q^2\right)\psi\left(q^6\right)\right]\\ &= a^2\left(q^4\right)+4qa\left(q^4\right)\psi\left(q^2\right)\psi\left(q^6\right)+4q^2\psi^2\left(q^2\right)\psi^2\left(q^6\right)\\ &\quad +16q^2\psi^2\left(q^2\right)\psi^2\left(q^6\right)+16q\psi\left(q^2\right)\psi\left(q^6\right)a\left(q^4\right)\\ &\quad +32q^2\psi^2\left(q^2\right)\psi^2\left(q^6\right).\end{aligned}$$

Extract the even powers of q, then replace q^2 with q. The result is

$$\sum_{n=0}^{\infty} s_{(3,3,1,1)}(8n)\,q^n = a^2\left(q^2\right)+4q\psi^2(q)\,\psi^2\left(q^3\right)+48q\psi^2(q)\,\psi^2\left(q^3\right).$$

From Equation (16), we have

$$\sum_{n=0}^{\infty} s_{(3,3,1,1)}(8n)\,q^n = \sum_{n=0}^{\infty} s_{(3,3,1,1)}(2n)\,q^n+48q\psi^2(q)\,\psi^2\left(q^3\right). \quad (18)$$

Apply Equation (12) to Equation (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} s_{(3,3,1,1)}(8n)\,q^n &= \sum_{n=0}^{\infty} s_{(3,3,1,1)}(2n)\,q^n+48q\left[q^{-\frac{3}{5}}\omega\left(q^{\frac{3}{5}}\right)\right]^2\left[q^{-\frac{1}{5}}\omega\left(q^{\frac{1}{5}}\right)\right]^2\\ &= \sum_{n=0}^{\infty} s_{(3,3,1,1)}(2n)\,q^n+48q^{-\frac{3}{5}}\omega^2\left(q^{\frac{3}{5}}\right)\omega^2\left(q^{\frac{1}{5}}\right)\\ &= \sum_{n=0}^{\infty} s_{(3,3,1,1)}(2n)\,q^n+48\sum_{n=0}^{\infty} c_{(3,3,1,1)}(n)\,q^{\frac{n-3}{5}}.\end{aligned}$$

Equate coefficients of q^n gives

$$s_{(3,3,1,1)}(8n) = s_{(3,3,1,1)}(2n)+48c_{(3,3,1,1)}(5n+3).$$

For $\lambda=(3,3,1,1)$, we have $m=4, i_1=2, i_2=0, i_3=2$. Hence

$$\beta_{(3,3,1,1)}=2^4+2^3\left(\binom{2}{4}+\binom{2}{2}\binom{0}{1}+\binom{2}{1}\binom{2}{1}\right)=48.$$

Thus, Theorem 1 for the partition $\lambda=(3,3,1,1)$ is proved. □

3. Conclusion

Relation between the number of representations of an integer n as a sum of squares induced by λ, $s_\lambda(n)$ and the number of representations of an integer n as a sum of centred pentagonal numbers induced by λ, $c_\lambda(n)$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ a partition of 8 is given by

$$s_\lambda(8n) - s_\lambda(2n) = \beta_\lambda c_\lambda(5n+3)$$

for all non-negative integers n where $\beta_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right)$ for i_j denote the number of parts of λ which are equal to j. There are 22 partitions of 8 to be considered. Therefore, we have proved the obtained relation for all partitions of 8 using the generating function method. This project can be extended to find the relation between $s_\lambda(n)$ and $c_\lambda(n)$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a partition of $k > 8$.

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